THE RANK-ONE LIMIT OF THE FOURIER-MUKAI TRANSFORM

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Abstract. We give a formula for the specialization of the Fourier-Mukai transform on a semi-abelian variety of torus rank 1.

1. Introduction

Let $\pi : \mathcal{X}^\star \to S$ be a semi-abelian variety of relative dimension g over the spectrum S of a discrete valuation ring R with algebraically closed residue field k such that the generic fibre X_η is a principally polarized abelian variety. We assume that $\mathcal{X}^\star$ is contained in a complete rank-one degeneration $\mathcal{X}$. In particular, the special fibre X_0 of $\mathcal{X}$ is a complete variety over k containing as an open part the total space of the $\mathbb{G}_m$-bundle associated to a line bundle $J \to B$ over a $g-1$-dimensional abelian variety B. The normalization $\nu : \mathbb{P} \to X_0$ of X_0 can be identified with the $\mathbb{P}^1$-bundle over B associated to J and X_0 is obtained by identifying the zero-section of $\mathbb{P} \cong B$ with the infinity-section of $\mathbb{P}$ by a translation. Moreover, X_0 is provided with a theta divisor that is the specialization of the polarization divisor on the generic fibre.

If c_η is an algebraic cycle on X_η we can take the Fourier-Mukai transform $\varphi_\eta := F(c_\eta)$ and consider the limit cycle (specialization) φ_0 of φ_η. A natural question is: What is the limit φ_0 of φ_η?

If $q : \mathbb{P} \to B$ denotes the natural projection of the $\mathbb{P}^1$-bundle, the Chow ring of $\mathbb{P}$ is the extension $\mathrm{CH}^*(B)[\eta]/(\eta^2 - \eta \cdot q^* c_1(J))$ with $\eta = c_1(O_{\mathbb{P}}(1))$. We consider now cycles with rational coefficients. We denote by c_0 the specialization of the cycle c_η on X_0. We can write c_0 as $\nu_*(\gamma)$ with $\gamma = q^*z + q^*w \cdot \eta$.

Theorem 1.1. *Let c_η be a cycle on X_η with $c_0 = \nu_*(q^*z + q^*w \cdot \eta)$. The limit φ_0 of the Fourier-Mukai transform $\varphi_\eta = F(c_\eta)$ is given by $\varphi_0 = \nu_*(q^*a + q^*b \cdot \eta)$ with*

$$a = F_B(w) + \sum_{n=0}^{2g-2} \sum_{m=0}^{n} \frac{(-1)^m}{(n+2)!} F_B[(z + w \cdot c_1(J)) \cdot c_1^m(J)] \cdot c_1^{n-m+1}(J)$$

and

$$b = \sum_{n=0}^{2g-2} \sum_{m=0}^{n} \frac{(-1)^m}{(n+2)!} F_B[(((-1)^{n+1} - 1)z - w \cdot c_1(J)) \cdot c_1^m(J)] \cdot c_1^{n-m}(J)\,,$$

where F_B is the Fourier-Mukai transform of the abelian variety B.

We denote algebraic equivalence by $\stackrel{a}{=}$. The relation $c_1(J)\stackrel{a}{=}0$ implies the following result.

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Theorem 1.2. *With the above notation the limit φ_0 satisfies*

$$\varphi_0 \stackrel{a}{=} \nu_*(q^*F_B(w) - q^*F_B(z)\cdot\eta)\,.$$

Note that this is compatible with the fact that for a principally polarized abelian variety A of dimension g the Fourier-Mukai transform satisfies $F_A \circ F_A = (-1)^g(-1_A)^*$.

Beauville introduced in [2] a decomposition on the Chow ring with rational coefficients of an abelian variety using the Fourier-Mukai transform. Theorem 1.2 can be used to deduce non-vanishing results for Beauville components of cycles on the generic fibre of a semi-abelian variety of rank 1; we refer to §8 for examples.

We prove the theorem by constructing a smooth model $\mathcal{Y}$ of $\mathcal{X}\times_S\mathcal{X}$ to which the addition map $\mathcal{X}^\star \times_S \mathcal{X}^\star \to \mathcal{X}^\star$ extends and by choosing an appropriate extension of the Poincaré bundle to $\mathcal{Y}$. The proof is then reduced to a calculation in the special fibre. We refer to Fulton's book [8] for the intersection theory we use. The theory in that book is built for algebraic schemes over a field. In our case we work over the spectrum of a discrete valuation ring. But as is stated in § 20.1 and 20.2 there, most of the theory in Fulton's book, including in particular the statements we use in this paper, is valid for schemes of finite type and separated over S. However, for us projective space denotes the space of hyperplanes and not lines, which conflicts with Fulton's book, but is in accordance with [10].

2. Families of abelian varieties with a rank one degeneration

We now assume that R is a complete discrete valuation ring with local parameter t, field of quotients K and algebraically closed residue field k. Suppose that $(\mathcal{X}^\star, \mathcal{L})$ is a semi-abelian variety over $S = \mathrm{Spec}(R)$ such that the generic fibre X_η is abelian and the special fibre X_0^* has torus rank 1; moreover, we assume that $\mathcal{L}$ is a cubical invertible sheaf (meaning that $\mathcal{L}$ satisfies the theorem of the cube, see [7], p. 2, 8) and L_η is ample. In particular, the special fibre of $\mathcal{X}^\star$ fits in an exact sequence

$$1 \to T_0 \to X_0^* \to B \to 0,$$

where B is an abelian variety over k and T_0 the multiplicative group $\mathbb{G}_m$ over k. The torus T_0 lifts uniquely to a torus T_i of rank 1 over $S_i = \mathrm{Spec}(R/(t^{i+1})$ in $X_i^* = \mathcal{X}^\star \times_S S_i$. The quotient X_i^*/T_i is an abelian variety B_i over S_i. The system $\{B_i\}_{i=1}^\infty$ defines a formal abelian variety which is algebraizable, so that we have an exact sequence of group schemes over S

$$1 \to T \to G \xrightarrow{\pi} \mathcal{B} \to 0,$$

cf. [F-C, p. 34]. We assume now that we are given a line bundle M on $\mathcal{B}$ defining a principal polarization $\lambda : B \to B^t$ and consider $\pi^*(M)$. This defines a cubical line bundle on G. The extension G is given by a homomorphism c of the character group $Z \cong \mathbb{Z}$ of T to $\mathcal{B}^t$. The semi-abelian group scheme dual to $\mathcal{X}^\star$ defines a similar extension

$$1 \to T^t \to G^t \to \mathcal{B}^t \to 0$$

and the polarization provides an isomorphism ϕ of the character group Z of T with the character group Z^t of T^t. Now the degenerating abelian variety (i.e. semi-abelian variety) $\mathcal{X}^\star$ over S gives rise to the set of degeneration data (cf. [7], p 51, Thm 6.2, or [1], Def. 2.3):

(i) an abelian variety $\mathcal{B}$ over S and a rank 1 extension G. This amounts to a S-valued point b of $\mathcal{B} = \mathcal{B}^t$.

(ii) a K-valued point of G lying over b.
(iii) a cubical ample sheaf L on G inducing the polarization on $\mathcal{B}$ and an action of $Z = Z^t$ on L_η.

A section $s \in \Gamma(G, L)$ can be written uniquely as $s = \sum_{\chi \in Z} \sigma_\chi(s)$, where $\sigma_\chi : \Gamma(G, L) \to \Gamma(\mathcal{B}, M_\chi)$ is a R-linear homomorphism and M_χ is the twist of M by χ: in fact $\pi_*(O_G) = \oplus_\chi O_\chi$ with O_χ the subsheaf consisting of χ-eigenfunctions. (We refer to [7], p. 43; note also the sign conventions there in the last lines.) We have now by the action

$$c^t(y)^* M \cong M_{\phi(y)} \cong M \otimes O_{\phi(y)}, \qquad y \in Z^t.$$

This satisfies $\sigma_{\chi+1}(s) = \psi(1)\tau(\chi)T_b^*(\sigma_\chi(s))$, where τ is given by a point of $G(K)$ lying over b and ψ is as in [7], p. 44. We refer to Faltings-Chai's theorem (6.2) of [7], p. 51 for the degeneration data.

The compactification $\mathcal{X}$ of $\mathcal{X}^\star$ is now constructed as a quotient of the action of Z^t on a so-called relatively complete model. Such a relatively complete model $\tilde{P}$ for G can be constructed here in an essentially unique way. If B is trivial (i.e. $\dim(B) = 0$) and if the torus is $T = \mathrm{Spec}(R[z, z^{-1}])$ it is given as the toroidal variety obtained by gluing the affine pieces

$$U_n = \mathrm{Spec}(R[x_n, y_n]), \qquad \text{with} \qquad x_n y_n = t$$

where $G \subset \tilde{P}$ is given by $x_n = z/t^n, \quad y_n = t^{n+1}/z$, cf. [13], also in [7], p. 306]. By glueing we obtain an infinite chain $\tilde{P}_0$ of $\mathbb{P}^1$'s in the special fibre. We can 'divide' by the action of Z^t; this is easy in the analytic case, more involved in the algebraic case, but amounts to the same, cf. [13], also [7], p. 55-56.

In the special fibre we find a rational curve with one ordinary double point. If instead we divide by the action of nZ^t for $n > 1$ we find a cycle consisting of n copies of $\mathbb{P}^1$.

In case the abelian part B is not trivial we take as a relatively complete model the contracted (or smashed) product $\tilde{P} \times^T G$ with $\tilde{P}$ the relatively complete model for the case that B is trivial. Call the resulting space $\tilde{P}$. Then $\tilde{P}$ corresponds by Mumford's [loc. cit., p 29] to a polyhedral decomposition of $Z^t \otimes \mathbb{R} = \mathbb{R}$ with Z^t the cocharacter group of T. Then we essentially divide through the action of Z^t or nZ^t as before and obtain a proper $\mathcal{X} \to S$.

We describe the central fibre X_0 of $\mathcal{X}$. Let b be the k-valued point of $B \cong B^t$ that determines the above $\mathbb{G}_m$-extension. If M denotes a line bundle defining the principal polarization of B we let M_b be the translation of M by b and we set $J = M \otimes M_b^{-1}$ and define the projective bundle $\mathbb{P} = \mathbb{P}(J \oplus \mathcal{O}_B)$ with projection $q : \mathbb{P} \to B$. The bundle $\mathbb{P}$ has two natural sections (with images) $\mathbb{P}_1$ and $\mathbb{P}_2$ corresponding to the projections $J \oplus \mathcal{O}_B \to J$ and $J \oplus \mathcal{O}_B \to \mathcal{O}_B$. We have $\mathcal{O}(\mathbb{P}_1) \cong \mathcal{O}(\mathbb{P}_2) \otimes q^* J$ and $\mathcal{O}(1) \cong \mathcal{O}(\mathbb{P}_1)$ with $\mathcal{O}(1)$ the natural line bundle on $\mathbb{P}$. We denote by $\overline{\mathbb{P}}$ the non-normal variety obtained by gluing the sections $\mathbb{P}_1$ and $\mathbb{P}_2$ under a translation by the point b. The singular locus of $\overline{\mathbb{P}}$ has support isomorphic to B. The line bundle $\tilde{L} = \mathcal{O}(\mathbb{P}_1) \otimes q^* M_b \cong \mathcal{O}(\mathbb{P}_2) \otimes q^* M$ descends to a line bundle $\overline{L}$ on $\overline{\mathbb{P}}$ with a unique ample divisor D, see [14]. The central family X_0 of the family $\pi : \mathcal{X} \to S$ is then equal to $\bar{\mathbb{P}}$. The cubical invertible sheaf $\mathcal{L}$ on $\mathcal{X}^\star$ extends (uniquely) to $\mathcal{X}$ and its restriction to the central fiber $\bar{\mathbb{P}}$ is the line bundle $\overline{L}$, see [15].

3. Extension of the addition map

The addition map $\mu : \mathcal{X}^\star \times_S \mathcal{X}^\star \to \mathcal{X}^\star$ of the semi-abelian scheme $\mathcal{X}^\star$ does not extend to a morphism $\mathcal{X} \times_S \mathcal{X} \to \mathcal{X}$, but it does so after a small blow-up of $\mathcal{X} \times_S \mathcal{X}$ as we shall see.

The degeneration data of $\mathcal{X}^\star$ defines (product) degeneration data for $\mathcal{X}^\star \times_S \mathcal{X}^\star$. Indeed, we can take the fibre product of the relatively complete model $\tilde{P}' = \tilde{P} \times_S \tilde{P}$ and this corresponds (e.g. via [13], Corollary (6.6)) to the standard polyhedral decomposition of $\mathbb{R}^2 = (Z^t \otimes \mathbb{R})^2$ by the lines $x = m$ and $y = n$ for $m, n \in \mathbb{Z}$. The special fibre of the model $\tilde{P}'$ is an infinite union of $\mathbb{P}^1 \times \mathbb{P}^1$-bundles over $B \times B$ glued along the fibres over 0 and ∞. The compactified model of $\mathcal{X} \times_S \mathcal{X}$ is obtained by taking the 'quotient' of $\tilde{P}'$ under the action of $Z^t \times Z^t$. This is not regular; for example the criterion of Mumford ([13], p. 29, point (D)]) is not satisfied. We can remedy this by subdividing. For example, by taking the decomposition of $\mathbb{R}^2$ given by the lines $x = m, y = n$ and $x + y = l$ for $m, n, l \in \mathbb{Z}$.

The special fibre of this model is an infinite union of copies of $\mathbb{P}^1 \times \mathbb{P}^1$-bundles over $B \times B$ blown up in the two anti-diagonal sections $(0, \infty) = \mathbb{P}_1 \times \mathbb{P}_2$ and $(\infty, 0) = \mathbb{P}_2 \times \mathbb{P}_1$. This is regular.

Both the polyhedral decompositions are invariant under the action of translations $(x, y) \mapsto (x + a, y + b)$ for fixed $a, b \in \mathbb{Z}$. This means that we can form the 'quotient' by $Z^t \times Z^t \cong \mathbb{Z}^2$ (or a subgroup $nZ^t \times nZ^t$) and obtain a completed semi-abelian abelian variety $\mathcal{Y}$ of relative dimension $2g$ over S. We denote by $\epsilon : \mathcal{Y} \to \mathcal{Y}' = \mathcal{X} \times_S \mathcal{X}$ the natural map. We shall write V for Y_0 and $\sigma : \tilde{V} \to V$ for its normalization. Then $\tilde{V}$ is an irreducible component of the special fibre of $\tilde{P}'$. We denote by $\tau : \tilde{V} \to \mathbb{P}^1 \times \mathbb{P}^1$ the blow up map and by E_{12} and E_{21} the exceptional divisors over the blowing up loci $\mathbb{P}_1 \times \mathbb{P}_2$ and $\mathbb{P}_2 \times \mathbb{P}_1$, respectively.

Now consider the addition map $\mu : \mathcal{X}^\star \times_S \mathcal{X}^\star \to \mathcal{X}^\star$ with $\mathcal{X}^\star$ as in the preceding section. This morphism is induces (and is induced by) by a map $\tilde{\mu} : G \times_S G \to G$. However, this map does not extend to a morphism of the relatively complete model $\tilde{P}'$ since the corresponding (covariant) map $(Z^t \otimes \mathbb{R})^2 \to (Z^t \otimes \mathbb{R})$ does not have the property that it maps cells to cells. After subdividing (by adding the lines $x + y = l$ with $l \in \mathbb{Z}$) this property is satisfied (cf. [11], Thm. 7, p. 25). This means that the map μ extends to $\tilde{\mu} : \tilde{P}' \to \tilde{P}$ for the polyhedral decomposition given by this subdivision. It is compatible with the action of $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ and hence descends to a morphism $\bar{\mu} : \mathcal{Y} \to \mathcal{X}$. We summarize:

Proposition 3.1. *The addition map of group schemes $\mu : \mathcal{X}^\star \times_S \mathcal{X}^\star \to X^\star$ extends to a morphism $\bar{\mu} : \mathcal{Y} \to \mathcal{X}$.*

In the next section we shall see that the change from the model $\mathcal{X} \times_S \mathcal{X}$ to $\mathcal{Y}$ is a small blow-up.

For later calculations we write down this map explicitly on the special fibre. We start with $g = 1$; then B is trivial and we may restrict the map to an irreducible component of the special fibre of the relatively complete model $\tilde{P} \times_S \tilde{P}$ and get the map $m : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ given by $((a : b), (a' : b')) \mapsto (aa' : bb')$. This is not defined in the points $(0, \infty)$ and $(\infty, 0)$. After blowing up these points (which corresponds exactly to the change from $\mathcal{X} \times_S \mathcal{X}$ to $\mathcal{Y}$) the rational map becomes a regular map $\tilde{m} : \tilde{V} \to \mathbb{P}^1$. It is defined by the two sections $\mathrm{prop}(p_1^*\{0\}) + \mathrm{prop}(p_2^*\{0\})$ and $\mathrm{prop}(p_1^*\{\infty\}) + \mathrm{prop}(p_2^*\{\infty\})$ of the linear system $|\tau^*(F_1 + F_2) - E_{12} - E_{21}|$ with F_1 and F_2 the horizontal and vertical fibre (with prop( ) meaning the proper

transform). The map $\tilde{m}$ descends to a map $\bar{m} : V \to \bar{\mathbb{P}}$ which is the restriction of the morphism $\bar{\mu} : \mathcal{Y} \to \mathcal{X}$ to the central fiber.

For the case that $g > 1$, note that we have the addition map $\mu_{\mathcal{X}^\star}$. Its restriction to the special fibre extends to a map of the relatively complete model and then restricts to a morphism $\tilde{m} : \tilde{V} \to \mathbb{P}$ that lifts the addition map μ_B of B. That means that it comes from a surjective bundle map (cf. [10], Ch. II, Prop. 7.12)

$$\delta : m_1^*(J \oplus \mathcal{O}) \cong (p_1^* q^* J \otimes p_2^* q^* J) \oplus \mathcal{O} \to N$$

with $m_1 := \mu_B \circ (q \times q) \circ \tau : \tilde{V} \to B$ and $N = \tau^*(p_1^*\mathcal{O}(\mathbb{P}_1) \otimes p_2^*\mathcal{O}(\mathbb{P}_1)) \otimes \mathcal{O}(-E_{12} - E_{21}))$ with $p_i : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ the ith projection. Then $m_1^*(J \oplus \mathcal{O})^\vee \otimes N$ is isomorphic to the direct sum of

$$\tau^* p_1^* \mathcal{O}(\mathbb{P}_i) \otimes \tau^* p_2^* \mathcal{O}(\mathbb{P}_i) \otimes \mathcal{O}(-E_{12} - E_{21}) \qquad (i = 1, 2).$$

The map δ is then given by the two sections $\mathrm{prop}(p_1^*\mathbb{P}_i) + \mathrm{prop}(p_2^*\mathbb{P}_i)$ of $\tau^* p_1^* \mathcal{O}(\mathbb{P}_i) \otimes \tau^* p_2^* \mathcal{O}(\mathbb{P}_i) \otimes \mathcal{O}(-E_{12} - E_{21})$ for $i = 1, 2$. The map $\tilde{m}$ descends to a map $\bar{m} : V \to \bar{\mathbb{P}}$ which is the restriction of the morphism $\bar{\mu} : \mathcal{Y} \to \mathcal{X}$ to the central fiber.

4. An explicit model of $\mathcal{Y}$

We now describe an explicit local construction of the model $\mathcal{Y}$ by blowing up the model $\mathcal{X} \times_S \mathcal{X}$. Let $A_S^{g+1} = \mathrm{Spec}(R[x_1, \ldots, x_{g+1}])$ denote affine S-space. In local coordinates, inside A_S^{g+1}, we may assume that the g-dimensional fibration $\pi : \mathcal{X}^\star \to S$ is given by the equation $x_1 x_2 = t$, where the coordinates $x_3, \ldots, x_{g+1}$ are not involved, see [14] p. 361-362. We may assume that the zero section of the family is defined by $x_i = 1$ for $i = 1, \ldots, g + 1$.

We form the fiber product $\pi : \mathcal{Y}' = \mathcal{X} \times_S \mathcal{X}$. We denote by T the support of the singular locus of X_0. The $2g + 1$ dimensional variety $\mathcal{Y}'$ is singular in the special fiber along $\Sigma = T \times_k T \cong B \times_k B$ of dimension $2g - 2$. The generic fiber Y'_η is the product $X_\eta \times_K X_\eta$ of the abelian variety X_η, while the zero fiber Y'_0 is singular. The local equations of $\mathcal{Y}'$ in a neighborhood of the singular locus of the family are given in our local coordinates by the system $x_1 x_2 = t$, $x'_1 x'_2 = t$. The singular locus Σ of $\mathcal{Y}'$ is given by the equations $x_1 = x_2 = x'_1 = x'_2 = t = 0$.

The above blow up $\epsilon : \mathcal{Y} \to \mathcal{Y}'$ is a small blow up and can be described directly as follows: we blow up $\mathcal{Y}'$ along its subvariety Π defined by $x_1 = x'_2 = 0$ (a 2-plane contained in the central fiber of $\mathcal{Y}'$). The proper transform $\mathcal{Y}$ of $\mathcal{Y}'$ is smooth. In local coordinates, the blow-up is given by the graph $\Gamma_\phi \subseteq Y' \times \mathbb{P}^1$ of the rational map $\phi : \mathcal{Y}' \longrightarrow \mathbb{P}^1$ given by $\phi(x_1, \ldots, x'_{g+1}, t) = (x_1 : x'_2)$. The equations of the graph $\Gamma_\phi \subseteq Y' \times \mathbb{P}^1 \subseteq A_S^{2(g+1)} \times_S \mathbb{P}^1_S$ are given by the system

$$x_1 x_2 = t,\ u x'_2 - v x_1 = 0,\ u x_2 - v x'_1 = 0\,,$$

where u, v are homogeneous coordinates on $\mathbb{P}^1$.

5. Extension of the Poincaré bundle

We denote by $j_0 : X_0 \hookrightarrow \mathcal{X}$ and $i_0 : Y_0 \hookrightarrow \mathcal{Y}$ the inclusions of the special fiber. Recall that we write V for Y_0 and $\tilde{V}$ for its normalization. We denote by $\mathcal{P}_\eta$ the Poincaré bundle on Y'_η and by P_B the Poincaré bundle on B.

Theorem 5.1. *The Poincaré bundle $\mathcal{P}_\eta$ has an extension $\mathcal{P}$ such that the pull back of $\mathcal{P}_0 := i_0^*\mathcal{P}$ to $\tilde{V}$ satisfies $\sigma^*\mathcal{P}_0 \cong \tau^*(q \times q)^* P_B \otimes \mathcal{O}(-E_{12} - E_{21})$.*

Proof. We have the following commutative diagram of maps

$$
\begin{tikzcd}
 & V \arrow[r, "\bar{m}"] & \overline{\mathbb{P}} \\
 & \tilde{V} \arrow[u, "\sigma"] \arrow[r, "\tilde{m}"] \arrow[d, "\tau"] & \mathbb{P} \arrow[u, "\nu"'] \arrow[d, equal] \\
\mathbb{P} \arrow[d, "q"] & \mathbb{P} \times \mathbb{P} \arrow[l, "p_i"'] \arrow[d, "q \times q"'] & \mathbb{P} \arrow[d, "q"'] \\
B & B \times B \arrow[l, "q_i"'] \arrow[r, "\mu_B"] & B
\end{tikzcd}
$$

Let $\mathcal{L}$ be the theta line bundle on the family $\mathcal{X}$ introduced in section 2. We define the extension of $\mathcal{P}^0$ by

$$\mathcal{P} := \bar{\mu}^* \mathcal{L} \otimes \rho_1^* \mathcal{L}^{-1} \otimes \rho_2^* \mathcal{L}^{-1},$$

where we denote by $\rho_1, \rho_2 : \mathcal{Y} \to \mathcal{X}$ the compositions of the natural projections ${\rho'}_i : \mathcal{Y}' \to \mathcal{X}$ with the blowing up map $\epsilon : \mathcal{Y} \to \mathcal{Y}'$ of section 4. We then have $\sigma^* \mathcal{P}_0 = \sigma^*(\bar{m}^* j_0^* \mathcal{L}) \otimes \sigma^* i_0^* \rho_1^* \mathcal{L}^{-1} \otimes \sigma^* i_0^* \rho_2^* \mathcal{L}^{-1}$. Now $\bar{m}^* j_0^* \mathcal{L} = \bar{m}^* \bar{L}$, so $\sigma^*(\bar{m}^* j_0^* \mathcal{L}) = \sigma^* \bar{m}^* \bar{L} = \tilde{m}^* \nu^* \bar{L} = \tilde{m}^*(\mathcal{O}(\mathbb{P}_1) \otimes q^* M_b)$. In view of $\mathcal{O}(\mathbb{P}_1) = \mathcal{O}(1)$ we have $\tilde{m}^* \mathcal{O}(\mathbb{P}_1) = N$, where N is the line bundle introduced at the end of section 3. We thus get

$$\tilde{m}^* \mathcal{O}(\mathbb{P}_1) = \tau^* p_1^* \mathcal{O}(\mathbb{P}_1) \otimes \tau^* p_2^* \mathcal{O}(\mathbb{P}_1) \otimes \mathcal{O}(-E_{12} - E_{21})$$

and $\tilde{m}^* q^* M_b = \tau^*(q \times q)^* \mu_B^* M_b$. On the other hand using the description of $\bar{L}$ in §2 we see

$$\begin{aligned} \sigma^*(i_0^* \rho_i^* \mathcal{L}) =& \tau^* p_i^* \nu^* \bar{L} = \tau^* p_i^* (\mathcal{O}(\mathbb{P}_1) \otimes q^* M_b) \\ =& \tau^* p_i^* \mathcal{O}(\mathbb{P}_1) \otimes \tau^* (q \times q)^* q_i^* M_b. \end{aligned}$$

and putting this together we find

$$\begin{aligned} \sigma^* \mathcal{P}_0 &= \tau^*(q \times q)^* (\mu_B^* M_b \otimes q_1^* M_b^{-1} \otimes q_2^* M_b^{-1}) \otimes \mathcal{O}(-E_{12} - E_{21}) \\ &= \tau^*(q \times q)^* P_B \otimes \mathcal{O}(-E_{12} - E_{21}). \end{aligned}$$

□

6. The basic construction

The fibration $\pi : \mathcal{Y} \to S$ is a flat map since $\mathcal{Y}$ is irreducible and S is smooth 1-dimensional, see [10], Ch. III, Proposition 9.7. The maps $\rho_i = \mathcal{Y} \to \mathcal{X}$, $i = 1, 2$, defined in the proof of Theorem 5.1, are flat maps too since they are maps of smooth irreducible varieties with fibers of constant dimension g, see e.g. [12], Corollary of Thm. 23.1.

We denote by Y_0 (resp. Y_η) the special fibre (resp. the generic fibre) and by $i_0 : Y_0 \to \mathcal{Y}$ (resp. $i_\eta : Y_\eta \to \mathcal{Y}$) the corresponding embedding. According to [8], Example 10.1.2., i_0 is a regular embedding. Similarly, $j_0 : X_0 \to \mathcal{X}$ is a regular

embedding. We consider the diagram

$$\begin{tikzcd} Y_0 \arrow[r, "i_0"] \arrow[d, "\pi_0"] & \mathcal{Y} \arrow[d, "\pi"] \\ \mathrm{Spec}(k) \arrow[r, "s"] & S \end{tikzcd}$$

Let $i_0^*: A_k(\mathcal{Y}) \to A_{k-1}(Y_0)$ be the Gysin map (see [8], Example 5.2.1). Since Y_0 is an effective Cartier divisor in $\mathcal{Y}$ the Gysin map i_0^* coincides with the Gysin map for divisors (see [8], Example 5.2.1 (a) and § 2.6).

We now consider specialization of cycles, see [8], § 20.3. Note that according to [8], Remark 6.2.1., in our case we have $s^! a = i_0^* a,\ a \in A_*(\mathcal{Y})$. If $\mathcal{Z}$ is a flat scheme over the spectrum of a discrete valuation ring S the specialization homomorphism $\sigma_Z : A_k(Z_\eta) \to A_k(Z_0)$ is defined as follows, see [8], pg. 399: If β_η is a cycle on Z_η we denote by β an extension of β_η in $\mathcal{Z}$ (e.g. the Zariski closure of β_η in $\mathcal{Z}$) and then $\sigma_Z(\beta_\eta) = i_0^*(\beta)$, where $i_0 : Z_0 \to \mathcal{Z}$ is the natural embedding.

Let c_η be a cycle on X_η and let $\varphi_\eta = F(c_\eta)$ be the Fourier-Mukai transform. It is defined by $F(c_\eta) = \rho_{2*}(e^{c_1(\mathcal{P}_\eta)} \cdot \rho_1^* c_\eta) \in A_*(X_\eta)$. Let $\sigma_X : A_k(X_\eta) \to A_k(X_0)$ be the specialization map. We have to determine $\sigma_X(F(c_\eta))$.

If β_η is a cycle on $A_k(Y_\eta)$ we have $\rho_{2*}\sigma_Y(\beta_\eta) = \sigma_X \rho_{2*}(\beta_\eta)$ by applying [8] Proposition 20.3 (a) to the proper map $\rho_2 : \mathcal{Y} \to \mathcal{X}$. By choosing $\beta_\eta = e^{c_1(\mathcal{P}_\eta)} \cdot \rho_1^* c_\eta$ we have

$$\sigma_X(F(c_\eta)) = \rho_{2*}\sigma_Y(e^{c_1(\mathcal{P}_\eta)} \cdot \rho_1^* c_\eta)\ . \tag{1}$$

Therefore, in order to compute $\sigma_X(\mathcal{F}(c_\eta))$ we have to identify $\sigma_Y(e^{c_1(\mathcal{P}_\eta)} \cdot \rho_1^* c_\eta)$. We take the extension $e^{c_1(\mathcal{P})}$ of $e^{c_1(\mathcal{P}_\eta)}$ and the extension of $\rho_1^* c_\eta$ given by $\rho_1^* c$, where c is the Zariski closure of c_η in $\mathcal{X}$. Since $i_\eta : Y_\eta \to \mathcal{Y}$ is an open embedding and hence a flat map of dimension 0, we have $i_\eta^*(e^{c_1(\mathcal{P})} \cdot \rho_1^* c) = e^{c_1(\mathcal{P}_\eta)} \cdot \rho_1^* c_\eta$, see [8], Proposition 2.3 (d). In other words, the cycle $e^{c_1(\mathcal{P})} \cdot \rho_1^* c$ extends the cycle $e^{c_1(\mathcal{P}_\eta)} \cdot \rho_1^* c_\eta$ and hence $\sigma_Y(e^{c_1(\mathcal{P}_\eta)} \cdot \rho_1^* c_\eta) = i_0^*(e^{c_1(\mathcal{P})} \cdot \rho_1^* c)$.

Now, for any k-cycle a on $\mathcal{Y}$ we have the identity

$$i_0^*(c_1(\mathcal{P}) \cdot a) = c_1(\mathcal{P}_0) \cdot i_0^*(a)$$

in $A_{k-2}(Y_0)$, where $\mathcal{P}_0 = i_0^* \mathcal{P}$ is the pull back of the line bundle and $i_0^* a$ the Gysin pull back to the divisor Y_0. This follows from applying the formula in [8], Proposition 2.6 (e) to $i_0 : Y_0 \to \mathcal{Y}$, with $D = Y_0$, $X = \mathcal{Y}$ and $L = \mathcal{P}$ the Poincaré bundle. Hence

$$\sigma_Y(e^{c_1(\mathcal{P}_\eta)} \cdot \rho_1^* c_\eta) = e^{c_1(\mathcal{P}_0)} \cdot i_0^*(\rho_1^* c)\ . \tag{2}$$

By the Moving Lemma (see [8], §11.4), we may choose the cycle c on the regular $\mathcal{X}$ such that it intersects the singular locus T of the central fiber properly. Since $T \subseteq X_0$ the cycle $c_0 = j_0^*(c)$ meets T properly by the following dimension argument. We have $\dim(c \cap T) = \dim(c_0 \cap T)$, hence

$$\begin{aligned} \dim(c_0 \cap T) &= \dim(c) + \dim(T) - \dim(X) \\ &= (\dim(c) - 1) + \dim(T) - (\dim(X) - 1) \\ &= \dim(c_0) + \dim(T) - \dim(X_0). \end{aligned}$$

Since T is of codimension 1 in $X_0 = \bar{\mathbb{P}}$, saying that c_0 meets T properly, is equivalent to saying that no component of c_0 is contained in T.

Lemma 6.1. *There exists a cycle γ on $\mathbb{P}$ with $c_0 = \nu_* \gamma$ that meets the sections $\mathbb{P}_i$ for $i = 1, 2$ properly.*

Proof. If T is the singular locus of $\bar{\mathbb{P}}$ and $A = \mathbb{P}_1 \cup \mathbb{P}_2$ its preimage in $\mathbb{P}$, then $\bar{\mathbb{P}} \backslash T \cong \mathbb{P} \backslash A$. We may assume that the cycle c_0 is irreducible and we consider the support of $c_0 \cap (\bar{\mathbb{P}} \backslash T)$ as a subset W of $\mathbb{P} \backslash A$. Its Zariski closure $\gamma = \bar{W}$ is an irreducible cycle on $\mathbb{P}$. Then $\nu_* \gamma$ is an irreducible cycle on $\bar{\mathbb{P}}$ since the map ν is a projective map. Also, $\nu_* \gamma \cap (\bar{\mathbb{P}} \backslash T) = c_0 \cap (\bar{\mathbb{P}} \backslash T)$, hence $\nu_* \gamma$ is the Zariski closure of $c_0 \cap (\bar{\mathbb{P}} \backslash T)$ and so, by the irreducibility, we have $\nu_* \gamma = c_0$. □

Lemma 6.2. *If $c_0 = \nu_* \gamma$, then we have $i_0^* \rho_1^* c = \sigma_*(\tau^*(p_1^* \gamma))$.*

Proof. We denote the restriction of ρ_i to the special fibre again by ρ_i. Then we have $i_0^* \rho_1^* c = \rho_1^* c_0$ since ρ_1 is a flat map and i_0, j_0 are regular embeddings (see [8], Theorem 6.2 (b) and Remark 6.2.1). We will use the following commutative diagram

$$\begin{tikzcd}
& \tilde{V} \arrow[r, "\sigma"] \arrow[dl, "\tau"'] \arrow[dd] & V \arrow[dd, "\rho_i"'] \arrow[dr, "\epsilon"] & \\
\mathbb{P} \times \mathbb{P} \arrow[dr, "p_i"'] & & & \overline{\mathbb{P}} \times \overline{\mathbb{P}} \arrow[dl, "\rho_i'"] \\
& \mathbb{P} \arrow[r, "\nu"] & \overline{\mathbb{P}} &
\end{tikzcd}$$

We may assume that c_0 and γ are irreducible k-cycles. We claim that $\rho_1^* c_0$ is irreducible. Indeed, the map ρ_1 is a flat map of relative dimension g. The cycle $\rho_1^* c_0$ is then a cycle of pure dimension $k + g$ and contains the proper transform of $(\rho_1')^* c_0$ and that is an irreducible cycle. Any other irreducible component of $\rho_1^* c_0$ must have support on the preimage of T. But since the cycle c_0 intersects T along a $k-1$-cycle, there is no irreducible component of $\rho_1^* c_0$ on the preimage of T. On the other hand, since γ meets the sections $\mathbb{P}_i$ properly, the cycle $\tau^* p_1^* \gamma$ is an irreducible cycle, and hence so is $\sigma_*(\tau^* p_1^* \gamma)$. But as $\rho_1^* c_0$ and $\sigma_*(\tau^* p_1^* \gamma)$ coincide outside the exceptional divisor of V, they have to coincide everywhere. □

Proposition 6.3. *We have $\sigma_X(\mathcal{F}(c_\eta)) = \rho_{2*}(e^{c_1(\mathcal{P}_0)} \cdot \sigma_*(\tau^* p_1^* \gamma))$.*

Proof. By equation (2) and Lemma 6.2 we have

$$\sigma_Y(e^{c_1(\mathcal{P}_\eta)} \cdot \rho_1^* c_\eta) = e^{c_1(\mathcal{P}_0)} \cdot \sigma_* \tau^*(p_1^* \gamma) \ . \tag{3}$$

The result follows from equation (1). □

In order to calculate the limit of the Fourier-Mukai transform we are thus reduced to a calculation in the special fibre.

7. A calculation in the special fibre - Proof of the main theorem

Recall the normalization map $\sigma : \tilde{V} \to V$. Suppose we have a cycle ρ on $\tilde{V}$ with $\sigma_* \rho = c_0$. We can consider the intersection $c_1(\mathcal{P}_0)^k \cdot c_0$, that is a successive intersection of a cycle with a Cartier divisor on the singular variety V. On the

other hand we have the cycle $\sigma_*(c_1(\sigma^*\mathcal{P}_0)^k \cdot \rho)$ and the projection formula ([8], Proposition 2.5 (c)) implies that

$$c_1(\mathcal{P}_0)^k \cdot c_0 = \sigma_*(c_1(\sigma^*\mathcal{P}_0)^k \cdot \rho).$$

Now we will use the following diagram of maps.

$$\begin{tikzcd}
 & & & \tilde{V} \arrow[r, "\sigma"] \arrow[d, "\tau"] & V & & \\
 & & & \mathbb{P}\times\mathbb{P} \arrow[dll, "p_1"'] \arrow[dl, "\alpha_2"] \arrow[dd, "q\times q"] \arrow[dr, "\alpha_1"'] \arrow[drr, "p_2"] & & & \\
\overline{\mathbb{P}} & \mathbb{P} \arrow[l, "\nu"] \arrow[dr, "q"'] & \mathbb{P}\times B \arrow[l, "\kappa_1"] \arrow[dr, "\beta_1"] & & B\times\mathbb{P} \arrow[r, "\kappa_2"'] \arrow[dl, "\beta_2"'] & \mathbb{P} \arrow[r, "\nu"'] \arrow[dl, "q"] & \overline{\mathbb{P}} \\
 & & B & B\times B \arrow[l, "q_1"] \arrow[r, "q_2"'] & B & &
\end{tikzcd}$$

Lemma 7.1. *Let x be a cycle on $B \times B$. Then the following holds.*

(1) $p_{2*}((q\times q)^* x) = 0$.
(2) $p_{2*}((q\times q)^* x \cdot p_1^*\eta) = q^* q_{2*} x$.

Proof. For (1) we observe that $p_{2*} = \kappa_{2*}\alpha_{1*}$, and $(q\times q)^* = \alpha_1^*\beta_2^*$ and $\alpha_{1*}\alpha_1^* = 0$. For (2) we use the identities

$$\begin{aligned}
p_{2*}((q\times q)^* x \cdot p_1^*\eta) &= p_{2*}(\alpha_2^*\beta_1^* x \cdot \alpha_2^*\kappa_1^*\eta) = p_{2*}\alpha_2^*(\beta_1^* x \cdot \kappa_1^*\eta) \\
&= \kappa_{2*}\alpha_{1*}\alpha_2^*(\beta_1^* x \cdot \kappa_1^*\eta) = \kappa_{2*}\beta_2^*\beta_{1*}(\beta_1^* x \cdot \kappa_1^*\eta) \\
&= \kappa_{2*}\beta_2^*(x \cdot \beta_{1*}\kappa_1^*\eta) = q^* q_{2*}(x \cdot q_1^* q_*\eta) = q^* q_{2*} x.
\end{aligned}$$

$\square$

Consider the following diagram of maps

$$\begin{tikzcd}
\mathbb{P}_i \arrow[d, "\lambda_i"'] & \mathbb{P}_i\times\mathbb{P}_j \arrow[d, "\lambda_{ij}"'] & E_{ij} \arrow[l, "\pi_{ij}"'] \arrow[d, "\epsilon_{ij}"'] \\
\mathbb{P} \arrow[d, "q"'] & \mathbb{P}\times\mathbb{P} \arrow[l, "p_1"'] \arrow[d, "q\times q"'] & \tilde{V} \arrow[l, "\tau"'] \arrow[d, "\sigma"'] \\
B & B\times B \arrow[l, "q_1"'] \arrow[d, "q_2"'] & V \\
 & B &
\end{tikzcd}$$

where p_i, q_i are the projections to the ith factor, π_{ij} the canonical map of the projective bundle E_{ij} and the maps λ_i, λ_{ij} and ϵ_{ij} the natural inclusions. The map $(q\times q)\circ\lambda_{ij}$ is an isomorphism.

By the adjunction formula, the normal bundles to $\mathbb{P}_1$, $\mathbb{P}_2$ are $N_{\mathbb{P}_1}(\mathbb{P}) = J$ and $N_{\mathbb{P}_2}(\mathbb{P}) = J^{-1}$. The exceptional divisors E_{12} and E_{21} are projective bundles over the blowing up loci $\mathbb{P}_i \times \mathbb{P}_j$. By identifying $\mathbb{P}_i \times \mathbb{P}_j$ with $B \times B$, via the map $(q\times q)\circ\lambda_{ij}$, we have $E_{12} = \mathbb{P}(q_1^* J^{-1} \oplus q_2^* J)$ and $E_{21} = \mathbb{P}(q_1^* J \oplus q_2^* J^{-1})$. We set

$\xi_{ij} = c_1(O(1))$ on E_{ij}. By standard theory [[10], ch. II, Theorem 8.24 (c)] we have $\epsilon_{ij}^* E_{ij} = -\xi_{ij}$.

We now introduce the notation

$$\gamma := c_1(J), \quad \gamma_i = q_i^* \gamma, \quad \eta_i = p_i^* \eta, \quad i = 1, 2.$$

Note that γ is algebraically equivalent to 0, but not rationally equivalent to 0. We have the quadratic relations

$$\xi_{ij}^2 + {\pi'}_{ij}^*(\gamma_i - \gamma_j) \cdot \xi_{ij} - {\pi'}_{ij}^*(\gamma_i \gamma_j) = 0,$$

where $\pi'_{ij} : E_{ij} \to B \times B$ is the natural map.

Lemma 7.2. *Suppose that ξ satisfies the relation $\xi^2 + (a-b)\xi - ab = 0$. Then, with $\phi_k = \sum_{m=0}^{k-1} (-1)^m a^m b^{k-1-m}$ we have $\xi^k = \phi_k \xi + ab\phi_{k-1}$ for any $k \geq 1$ (where we put $\phi_0 = 0$).*

Proof. Assuming by induction that $\xi^k = \phi_k \xi + ab\phi_{k-1}$ we find

$$\xi^{k+1} = \phi_k \xi^2 + ab\phi_{k-1}\xi = ((b-a)\phi_k + ab\phi_{k-1})\xi + ab\phi_k,$$

so the result follows by induction from the recurrence $\phi_{k+1} = (b-a)\phi_k + ab\phi_{k-1}$ that can be left to the reader. □

Applying the above for the classes ξ_{ij} of the bundles E_{ij}, considered as bundles over $B \times B$ via the isomorphism $(q \times q) \circ \lambda_{ij}$, we get, by choosing

$$\phi_k = \sum_{m=0}^{k-1} (-1)^m \gamma_1^m \gamma_2^{k-1-m},$$

that

$$\begin{aligned} \xi_{12}^k &= {\pi'}_{12}^* \phi_k \cdot \xi_{12} + {\pi'}_{12}^*(\gamma_1 \gamma_2 \phi_{k-1}), \\ \xi_{21}^k &= (-1)^{k+1} {\pi'}_{21}^* \phi_k \cdot \xi_{21} + (-1)^k {\pi'}_{21}^*(\gamma_1 \gamma_2 \phi_{k-1}) . \end{aligned}$$

We view now the bundles E_{ij} as bundles over $\mathbb{P}_i \times \mathbb{P}_j$ and, for any $k \geq 0$, we write $\xi_{ij}^k = \pi_{ij}^* A_{ij}(k) \xi_{ij} + \pi_{ij}^* B_{ij}(k)$, for some cycles $A_{ij}(k)$, $B_{ij}(k)$ on $\mathbb{P}_i \times \mathbb{P}_j$. By the above relations we have

$$(q \times q)_* \lambda_{ij*} A_{ij}(k) = (-1)^{(k+1)j} \phi_k \, .$$

Lemma 7.3. *We have*

$$\lambda_{ij*} A_{ij}(k) = (-1)^{(k+1)j} [(q \times q)^* \phi_k \cdot \eta_1 \eta_2 - (q \times q)^* (\phi_k \, \gamma_j) \cdot \eta_i] \, .$$

Proof. We let $\psi_{ij} = (q \times q) \circ \lambda_{ij} : \mathbb{P}_i \times \mathbb{P}_j \to B \times B$ be the natural isomorphism. We then have the identity

$$\lambda_{ij*} A_{ij}(k) = \lambda_{ij*}(\psi_{ij}^* \psi_{ij*} A_{ij}(k)) = (q \times q)^* \psi_{ij*} A_{ij}(k) \cdot \lambda_{ij*} 1_{\mathbb{P}_i \times \mathbb{P}_j} .$$

But $\lambda_{ij*} 1_{\mathbb{P}_i \times \mathbb{P}_j} = p_1^* \mathbb{P}_i \cdot p_2^* \mathbb{P}_j = \eta_i(\eta_j - p_j^* q^* \gamma) = \eta_1 \eta_2 - \eta_i \cdot (q \times q)^* \gamma_j$ and the result follows. □

Lemma 7.4. *For a cycle class $x = q^* z + q^* w \cdot \eta$ on $\mathbb{P}$ the cycle class $\tau_*(\tau^* p_1^* x \cdot (E_{12}^k + E_{21}^k))$ for $k \geq 1$ is given by*

$$\begin{aligned} &\sum_{m=0}^{k-2} (-1)^m \{ (q \times q)^* q_1^* [(((-1)^{k+1} - 1) z + (-1)^{k+1} w \gamma) \, \gamma^m] \cdot \eta_1 \eta_2 \\ &+ (-1)^k (q \times q)^* q_1^* [(z + w\gamma) \, \gamma^m] \cdot \eta_1 \cdot p_2^* q^* \gamma + (q \times q)^* q_1^* (z \gamma^{m+1}) \cdot \eta_2 \} \cdot p_2^* q^* \gamma^{k-2-m} . \end{aligned}$$

Note that for $k=1$ the above sum is zero.

Proof. Since $\epsilon_{ij}^* E_{ij} = -\xi_{ij}$ we have $E_{ij}^k = (-1)^{k-1}\epsilon_{ij*}\xi_{ij}^{k-1}$. Therefore

$$\begin{aligned}\tau_*(\tau^* p_1^* x \cdot E_{ij}^k) =&(-1)^{k-1} p_1^* x \cdot \tau_* \epsilon_{ij*} \xi_{ij}^{k-1} \\ =&(-1)^{k-1} p_1^* x \cdot \lambda_{ij*}\pi_{ij*}(\pi_{ij}^* A_{ij}(k-1)\xi_{ij} + \pi_{ij}^* B_{ij}(k-1)) \\ =&(-1)^{k-1} p_1^* x \cdot \lambda_{ij*} A_{ij}(k-1)\end{aligned}$$

since $\pi_{ij*}\xi_{ij} = 1_{\mathbb{P}_i \times \mathbb{P}_j}$. Note that since $A_{ij}(0)=0$ the above calculation shows that $\tau_*(\tau^* p_1^* x \cdot E_{ij}) = 0$. By Lemma 7.3 and by using the relation

$$p_1^* x = (q\times q)^* q_1^* z + (q\times q)^* q_1^* w \cdot \eta_1,$$

we have

$$\begin{aligned}\tau_*(\tau^* p_1^* x \cdot E_{ij}^k) =&(-1)^{k(j+1)+1}((q\times q)^* q_1^* z + (q\times q)^* q_1^* w \cdot \eta_1) \\ &\cdot [(q\times q)^* \phi_{k-1} \cdot \eta_1\eta_2 - (q\times q)^*(\phi_{k-1}\gamma_j)\cdot \eta_i]\end{aligned}$$

and this equals

$$\begin{aligned}&(-1)^{k(j+1)+1}[(q\times q)^*(q_1^* z\cdot \phi_{k-1})\cdot \eta_1\eta_2 - (q\times q)^*(q_1^* z \cdot \phi_{k-1}\gamma_j)\cdot \eta_i \\ &+ (q\times q)^*(q_1^* w\cdot \phi_{k-1})\cdot \eta_1^2\eta_2 - (q\times q)^*(q_1^* w\cdot \phi_{k-1}\gamma_j)\cdot \eta_1\eta_i]\end{aligned}$$

We then have, by using the formula $\eta^2 = q^*\gamma\cdot\eta$, that

$$\begin{aligned}\tau_*(\tau^* p_1^* x \cdot E_{12}^k) = (-1)^{k+1}&[(q\times q)^*(q_1^*(z+w\gamma)\cdot\phi_{k-1})\cdot \eta_1\eta_2 \\ &-(q\times q)^*(q_1^*(z+w\gamma)\cdot \phi_{k-1})\cdot\eta_1\cdot p_2^* q^*\gamma]\end{aligned}$$

and

$$\tau_*(\tau^* p_1^* x \cdot E_{21}^k) = -(q\times q)^*(q_1^* z\cdot\phi_{k-1})\cdot\eta_1\eta_2 + (q\times q)^*(q_1^*(z\,\gamma)\cdot\phi_{k-1})\cdot\eta_2.$$

Using $\phi_{k-1} = \sum_{m=0}^{k-2}(-1)^m\gamma_1^m\cdot\gamma_2^{k-2-m}$ we deduce the proposition. □

We state now the basic result of this section.

Proposition 7.5. *Let z, w be cycles on B. Then we have*

$$p_{2*}\tau_*(e^{c_1(\sigma^*\mathcal{P}_0)}\cdot\tau^*(p_1^*(q^*z + q^*w\cdot\eta)) = q^*a + q^*b\cdot\eta,$$

with a and b as in Theorem 1.1.

Proof. We put $x = q^*z + q^*w\cdot\eta$. We want to calculate

$$p_{2*}\tau_*(e^{\tau^*(q\times q)^* c_1(P_B) - E_{12} - E_{21}}\cdot\tau^*(p_1^* x))$$

which equals

$$p_{2*}(e^{(q\times q)^* c_1(P_B)}\cdot\tau_*(e^{-E_{12}-E_{21}}\cdot\tau^* p_1^* x)).$$

Since $E_{12}\cdot E_{21} = 0$ we have

$$e^{-E_{12}-E_{21}} = 1 + \sum_{k=1}^{2g}\frac{(-1)^k}{k!}\,(E_{12}^k + E_{21}^k)$$

and so $\tau_*(e^{-E_{12}-E_{21}}\cdot\tau^* p_1^* x)$ equals

$$p_1^* x + \sum_{k=1}^{2g}\frac{(-1)^k}{k!}\,\tau_*[\tau^* p_1^* x\cdot(E_{12}^k + E_{21}^k)].$$

We have

$$\begin{aligned} p_{2*}((q\times q)^* e^{c_1(P_B)}\cdot p_1^* x) &= p_{2*}(e^{(q\times q)^* c_1(P_B)}\cdot p_1^*(q^*z+q^*w\,\eta)) \\ &= p_{2*}((q\times q)^*(e^{c_1(P_B)}q_1^* z)+(q\times q)^*(e^{c_1(P_B)}q_1^* w)\,p_1^*\eta) \\ &= 0+q^*q_{2*}(e^{c_1(P_B)}q_1^* w)=q^*F_B(w) \end{aligned}$$

by Lemma 7.1. Combining the above with Lemma 7.4 we find that

$$p_{2*}\tau_*(e^{\tau^*(q\times q)^*c_1(P_B)-E_{12}-E_{21}}\cdot \tau^*(p_1^*x))$$

is the sum of the four terms: the first is $q^*F_B(w)$, the second is

$$\sum_{k=2}^{2g}\sum_{m=0}^{k-2}\frac{(-1)^{k+m}}{k!}\{p_{2*}[(q\times q)^*[e^{c_1(P_B)}q_1^*[(((-1)^{k+1}-1)z+ (-1)^{k+1}w\gamma)\,\gamma^m]]\cdot\eta_1]\}\cdot\eta\cdot q^*\gamma^{k-2-m},$$

the third term is

$$\sum_{k=2}^{2g}\sum_{m=0}^{k-2}\frac{(-1)^{m}}{k!}\{p_{2*}[(q\times q)^*[e^{c_1(P_B)}q_1^*[(z+w\gamma)\,\gamma^m]]\cdot\eta_1]\}\cdot q^*\gamma^{k-1-m},$$

and finally the fourth is

$$\sum_{k=2}^{2g}\sum_{m=0}^{k-2}\frac{(-1)^{k+m}}{k!}\{p_{2*}[(q\times q)^*[e^{c_1(P_B)}q_1^*(z\gamma^{m+1})]]\}\cdot\eta\cdot q^*\gamma^{k-2-m}\,.$$

By applying now Lemma 7.1 and by making the substitution $n=k-2$ we get the desired expression. □

Corollary 7.6. *Let z, w be cycles on B. Then in algebraic equivalence we have*

$$p_{2*}\tau_*(e^{c_1(\sigma^*\mathcal{P}_0)}\cdot\tau^*(p_1^*(q^*z+q^*w\cdot\eta))\overset{a}{=}q^*F_B(w)-q^*F_B(z)\cdot\eta.$$

Proof. Indeed, since $c_1(J)\overset{a}{=}0$ it is clear that $a\overset{a}{=}F_B(w)$ and $b\overset{a}{=}-q^*F_B(z)$ since the only non zero term of the sum corresponds to $m=0,\ n=0$. □

We conclude now with the proof of the basic Theorem 1.1 and Theorem 1.2:

Proof. By Proposition 6.3 we have $\varphi_0=\sigma_X F(c_\eta)=\rho_{2*}(e^{c_1(\mathcal{P}_0)}\cdot\sigma_*(\tau^*p_1^*\gamma))$. By the projection formula we have $e^{c_1(\mathcal{P}_0)}\cdot\sigma_*(\tau^*p_1^*\gamma)=\sigma_*(e^{c_1(\sigma^*\mathcal{P}_0)}\cdot\tau^*p_1^*\gamma)$. Observe now that $\rho_2\circ\sigma=\nu\circ(p_2\circ\tau):\tilde{V}\to\bar{\mathbb{P}}$, see the diagram in the proof of Lemma 6.2. The proof then follows from Proposition 7.5 and Corollary 7.6. □

8. Applications

Let $\mathcal{X}\to S$ be a completed rank-one degeneration as described in §2. According to Beauville [2] we have a decomposition of $CH^i_{\mathbb{Q}}(X_\eta)$ into subspaces which are eigenspaces for the action of the integers on X_η:

$$A^i_{\mathbb{Q}}(X_\eta)=\oplus_j A^i_{(j)}(X_\eta)$$

such that $n^*(x)=n^{2i-j}\,x$ for $x\in A^i(X_\eta)$. (Beauville works over $\mathbb{C}$, but his proof does not use more than the Fourier-Mukai transform which works over the residue field of η.) The multiplication map n acts as multiplication by n^{2i} on homology and therefore all cycles in $A^i_{(j)}(X_\eta)$ are homologically trivial for $j\neq 0$. Since under

the Fourier-Mukai transform we have $F(A^i_{(j)}(X_\eta)) = A^{g-i+j}_{(j)}(X_\eta)$, the elements of A^i that lie in $A^i_{(j)}$ can be characterized by their codimension (namely $g-i+j$).

Suppose now that $c = \sum c^{(j)} \in A^i(X_\eta)$ with $c^{(j)} \in A^i_{(j)}(X_\eta)$, where the decomposition corresponds to $\varphi := F(c) = \sum \varphi^{(j)}$ with $\varphi^{(j)} \in A^{g-i+j}(X_\eta)$.

Theorem 8.1. *Let $c = c_\eta = \sum c^{(j)} \in A^i(X_\eta)$ with $c^{(j)} \in A^i_{(j)}(X_\eta)$ such that $\varphi_0^{(j)} \neq 0$, where φ_0 is the specialization and $\varphi_0^{(j)}$ the codimension $g-i+j$-part of φ_0. Then $c^{(j)} \neq 0$.*

Proof. The specialization map preserves the codimension of cycles. Therefore, if $c^{(j)} = 0$ then $\varphi^{(j)} = 0$, hence $\varphi_0^{(j)} = 0$ and this contradicts our assumption. □

This theorem, which holds as well for cycles modulo algebraic equivalence, can be used to prove non-vanishing results for cycles. For the rest of this section we work modulo algebraic equivalence. For example, consider a threefold $\mathcal{Z}/S$ such that Z_η is a smooth cubic threefold and Z_0 is a generic nodal cubic threefold. The genericity assumption means that the corresponding canonical genus 4 curve C in $\mathbb{P}^3$ which is used to construct the Fano threefold, see e.g. [9] Section 2, is a generic curve and hence we may assume by Ceresa's result [4] that the class $C^{(1)}$ does not vanish in the Jacobian B of the curve C. Since C is a trigonal curve we have by [6] that $C^{(j)} \overset{a}{=} 0$ for $j \geq 2$. Hence the Beauville decomposition of C is $[C] \overset{a}{=} C^{(0)} + C^{(1)}$ with $F_B(C^{(0)}) \in A^1_{(0)}(B)$ and $F_B(C^{(1)}) \in A^2_{(1)}(B)$.

The Picard variety $\mathcal{X}/S$ of $\mathcal{Z}$ defines a principally polarized semi-abelian variety with central fibre a rank-one extension of the Jacobian B of the curve C, see [9], Corollary 6.3 and Section 10. The principal polarization on X_η is induced by a geometrically defined divisor Θ. Let Σ be the Fano surface of lines in Z_η. If $s \in \Sigma$ we denote by l_s the corresponding line in Z_η. For each $s \in S$ we have the divisor

$$D_s = \{s' \in S,\ l_{s'} \cap l_s \neq \emptyset\}$$

on S as defined in [5]. We then have a natural map

$$\Sigma \to \mathrm{Pic}^0(\Sigma), \qquad s \mapsto D_s - D_{s_0},$$

with $s_0 \in \Sigma$ a base point. It is well known that the cohomology class of Σ in $\mathrm{Pic}^0(\Sigma)$ is equal to that of the cycle $\Theta^3/3!$, see [5]. By [2], Propositions 3 and 4, we have that $A^3_{(j)}(X_\eta) = 0$ for $j < 0$ and $A^5_{(j)}(X_\eta) = 0$ for $j \neq 0$ in algebraic equivalence. We have therefore the decomposition

$$[\Sigma] \overset{a}{=} \Sigma^{(0)} + \Sigma^{(1)} + \Sigma^{(2)} \qquad \text{with } \Sigma^{(j)} \in A^3_{(j)}.$$

Indeed, $\Sigma^{(j)} \in A^3_{(j)}(X_\eta)$, hence $F(\Sigma^{(j)}) \in A^{2+j}_{(j)}(X_\eta)$ which is zero for $j \geq 3$. Now we show that $\Sigma^{(1)} \overset{a}{\neq} 0$, and we thus obtain a cycle which is homologically but not algebraically equivalent to zero. Since $\Theta \in A^1_{(0)}(X_\eta)$ this implies that Σ is homologically, but not algebraically equivalent to $\Theta^3/3!$.

We denote by $\mathcal{X}$ the completed rank one degeneration of X_η. The class $[\Sigma]$ degenerates to a cycle $[\Sigma_0] = \nu_*(\gamma)$ on the central fiber X_0 of class

$$\gamma \overset{a}{=} q^*[C] + \frac{1}{2}\, q^*[C * C] \cdot \eta,$$

where $C * C$ is the Pontryagin product, see [9], Propositions 10.1 and 8.1. In order to see that $\Sigma^{(1)} \overset{a}{\neq} 0$ it suffices by Theorem 8.1 to show that $\varphi_0^{(1)} \overset{a}{\neq} 0$ with φ_0 the limit of the Fourier-Mukai transform. By Theorem 1.2, we have

$$\varphi_0 \overset{a}{=} \nu_*(\frac{1}{2}\, q^*[F_B(C) \cdot F_B(C)] - q^* F_B(C) \cdot \eta),$$

hence

$$\varphi_0^{(1)} \overset{a}{=} \nu_*(q^*[F_B(C^{(0)}) \cdot F_B(C^{(1)})] - q^* F_B(C^{(1)}) \cdot \eta).$$

Since $C^{(1)} \overset{a}{\neq} 0$ we conclude that $\varphi_0^{(1)} \overset{a}{\neq} 0$, and this implies the result.

By using the specialization of the Fourier-Mukai transform we can deduce the specialization of the Beauville decomposition. We do this working modulo algebraic equivalence.

Proposition 8.2. *Let $c = c_\eta \in A^i(X_\eta)$ with specialization $c_0 = \nu_*(q^*z + q^*w \cdot \eta)$, where $z \in A^i(B)$ and $w \in A^{i-1}(B)$. Let $c = \sum c^{(j)}$ with $c^{(j)} \in A^i_{(j)}(X_\eta)$, and let $z = \sum z^{(j)}$ with $z^{(j)} \in A^i_{(j)}(B)$ and $w = \sum w^{(j)}$ with $w^{(j)} \in A^{i-1}_{(j)}(B)$ be the Beauville decompositions. If $c_0^{(j)}$ is the specialization of $c^{(j)}$, then*

$$c_0^{(j)} \overset{a}{=} \nu_*(q^* z^{(j)} + q^* w^{(j)} \cdot \eta)\,.$$

Proof. By the proof of the main theorem in [2], the component $c^{(j)}$ is defined as $(-1)^g F((-1)^* \phi^{(j)})$ with $\phi^{(j)} \in A^{g-i+j}(X_\eta)$ (notation as above). The inversion on X_η leaves the cell decomposition of the toroidal compactification invariant and hence extends naturally to X_0. So $c_0^{(j)}$ equals $(-1)^g F((-1)^* \phi_0^{(j)})$ with $\phi_0^{(j)} \in A^{g-i+j}(X_0)$. Therefore, by Theorem 1.2, we have

$$\begin{aligned} c_0^{(j)} &\overset{a}{=} (-1)^g F((-1)^* \nu_*(q^* F_B(w^{(j)}) - q^* F_B(z^{(j)}) \cdot \eta)) \\ &\overset{a}{=} (-1)^{g+j}(-1)^{g-1+j} \nu_*(-q^* z^{(j)} - q^* w^{(j)} \cdot \eta) = \nu_*(q^* z^{(j)} + q^* w^{(j)} \cdot \eta)\,. \end{aligned}$$

□

For example, let $\mathcal{C} \to S$ be a genus g curve with C_η a smooth curve and C_0 a one-nodal curve with normalization $\tilde{C}_0$. Let p be the node of C_0 and $x_1,\ x_2$ the points of $\tilde{C}_0$ lying over p. The compactified Jacobian $\mathcal{X} = \overline{P_{\mathcal{C}/S}}$ is then a complete rank one degeneration with central fiber the $\mathbb{P}^1$-bundle over $\mathrm{Pic}^0(\tilde{C}_0)$ associated to the line bundle $J = O(x_1 - x_2)$. Let $\bar{u} : \mathcal{C} \to \mathcal{X}$ be the compactified Abel-Jacobi map and let $c_\eta = [\bar{u}(C_\eta)]$. The cycle c_η specializes then to the cycle $c_0 = [\bar{u}(C_0)]$ with $c_0 \overset{a}{=} \nu_*(q^*[\mathrm{pt}] + q^* \tilde{c}_0 \cdot \eta)$, where $[\mathrm{pt}]$ is the class of a point and $\tilde{c}_0$ is the class of the Abel-Jacobi image of the smooth curve $\tilde{C}_0$ in $\mathrm{Pic}^0(\tilde{C}_0)$, see e.g. [9], Proposition 7.1. By Proposition 8.2 we have then

$$c_0^{(j)} \overset{a}{=} \begin{cases} q^* \tilde{c}_0^{(j)} \cdot \eta, & j \neq 0\,, \\ q^*[\mathrm{pt}] + q^* \tilde{c}_0^{(0)} \cdot \eta, & j = 0\,. \end{cases}$$

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